JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 2, March–April 2002

Motion Control of a Nonholonomic System Based on the Lyapunov Control Method

Kazuo Tsuchiya,* Takateru Urakubo,† and Katsuyoshi Tsujita‡
Kyoto University, Kyoto 606-8501, Japan

A new design method of a feedback controller for nonholonomic systems based on Lyapunov control is presented. In Lyapunov control, the control input is obtained by multiplying the gradient vector of the Lyapunov function by a tensor. The main contribution of our method is that this tensor is composed of two components, one of which is a negative definite symmetric tensor and the other of which is an asymmetric one. As a result, the goal point in the state space of the controlled system becomes the only globally stable equilibrium point, and exponential convergence to the goal point can be achieved. The proposed method is applied to a two-wheeled mobile robot, and the effectiveness is confirmed by numerical simulations.

Introduction

THE motion control of a nonholonomic system has been focused on by many researchers. The motion control of a two-wheeled mobile robot is a typical terrestrial application, whereas typical applications in space include the attitude control of a rigid spacecraft with two reaction wheels and the motion control of a planar space manipulator composed of three links. In nonholonomic systems, there exist constraints that are not integrable, for example, a no-slip condition for the wheels of the wheeled vehicle, the law of conservation of angular momentum for a space robot, and so on. Although this class of nonlinear systems are not controllable locally, they may be controlled globally by exploiting the constraints. They can not be controlled with the method of linear control theory, however, because they are not exactly linearizable.[1] Moreover, they can not be stabilized to an equilibrium point by any smooth state feedback control, even if they are controllable globally.[2]

Therefore, it is difficult to design a feedback controller that stabilizes a nonholonomic system to an equilibrium point, and the research on this problem has been extensive. The controllers that have been proposed thus far are classified as time-varying controllers[3–6] and discontinuous time-invariant controllers.[7–9] Time-varying controllers were originated by Samson.[3] Pomet proposed a method of designing this type of controller by using a time-varying Lyapunov function.[4] Their controllers are smooth time-varying controllers, but the rates of convergence are not exponential. Sørdalen and Egeland[5] and M'Closkey and Murray[6] proposed nonsmooth time-varying controllers that provide exponential rates of convergence.

On the other hand, discontinuous time-invariant feedback controllers have also been proposed, such as the class of controllers based on the idea of sliding mode control, proposed by Khennouf and Canudas de Wit.[7] They designed the controller by combining a linear feedback law and a discontinuous feedback law. The discontinuous feedback law stabilizes the invariant manifolds formed by the linear feedback law.[7] Lafferriere and Sontag proposed a method of designing a controller by using a piecewise smooth Lyapunov function.[10] Astolfi proposed a method of designing a controller by transforming an original system through a nonsmooth coordinate transformation and designing a smooth, time-invariant controller for the transformed system.[8,9] The controllers designed in Refs. 7 and 9 provide exponential rates of convergence.

In this paper, we consider a three-dimensional two-input nonholonomic system that has no drift and propose a method of control that is based on Lyapunov control. In Lyapunov control, the input is constructed by multiplying the gradient vector of the Lyapunov function by a negative definite symmetric tensor. However, nonholonomic systems cannot converge to the desired point with this type of input because there exist stable equilibrium points other than the desired point. On the other hand, if the input is constructed by multiplying the gradient vector of the Lyapunov function by an asymmetric tensor, all of the equilibrium points except the desired point may become unstable, whereas the value of the Lyapunov function is kept constant. In this paper, we propose a method of designing a controller based on Lyapunov control; the input is constructed by multiplying the gradient vector of the Lyapunov function by a tensor that is composed of a negative definite symmetric tensor and an asymmetric tensor. The designed controller is a discontinuous time-invariant state feedback one. We call this method of control an extended Lyapunov control method. A nonholonomic system with this type of input converges to the desired point, as long as it is controllable.

This paper is organized as follows: First, the type of system that we consider in this paper is expressed in terms of differential geometry, and the basic equations for control are derived. Then we explain the controller using an extended Lyapunov control method and investigate the behavior of the controlled system. Finally, the controller is applied to an example, a two-wheeled mobile robot, and the behavior of the controlled system is examined by numerical simulations.

Formulation of the System

First, we set a three-dimensional state space, $z = [z_1, z_2, z_3]^T$, and assume a Pfaffian equation on the space. The Pfaffian equation is generally expressed as follows:

$$\omega = f_1(z)dz_1 + f_2(z)dz_2 + f_3(z)dz_3 = 0 \tag{1}$$

where $f_i(z)$ is a function of z. We make the following assumptions:

1) We assume that f_i is finite.
2) We assume that $\partial f_i/\partial z_j$, the derivative of f_i for z_j, exists and is finite.
3) We assume that one of f_1, f_2, or f_3 is not zero at any point. Therefore, we set $f_3 \equiv -1$ in this paper, without loss of generality.
4) We assume that $\dot{z}_1$ and $\dot{z}_2$ can be taken as the inputs for control.

Then, the Pfaffian equation is rewritten as follows:

$$\omega = f_1(z)dz_1 + f_2(z)dz_2 - dz_3 = 0 \tag{2}$$

The distribution μ for this Pfaffian equation is the tangent space that is composed of the vector fields X_1 and X_2,

$$X_1 = \frac{\partial}{\partial z_1} + f_1\frac{\partial}{\partial z_3}, \qquad X_2 = \frac{\partial}{\partial z_2} + f_2\frac{\partial}{\partial z_3} \tag{3}$$

Received 13 December 1999; revision received 10 April 2001; accepted for publication 22 July 2001.

*Professor, Department of Aeronautics and Astronautics.
†Doctoral Candidate, Department of Aeronautics and Astronautics.
‡Graduate Research Assistant, Department of Aeronautics and Astronautics.

Lie bracket $[X_1 X_2]$ is computed as

$$[X_1 X_2] = X_1 X_2 - X_2 X_1 = L(z)\frac{\partial}{\partial z_3} \tag{4}$$

where

$$L(z) = -\left(\frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1} + f_2\frac{\partial f_1}{\partial z_3} - f_1\frac{\partial f_2}{\partial z_3}\right) \tag{5}$$

The condition for the distribution μ not to involve the Lie bracket $[X_1 X_2]$ is that

$$L(z) \neq 0 \tag{6}$$

This is the condition of controllability. When this condition is satisfied at all points, the Pfaffian form ω is converted to a normal form using a certain coordinate transformation[11]:

$$\omega = dz_3 - z_2 dz_1 \tag{7}$$

Therefore, the system can be expressed by a chained form. In this paper, we do not assume that condition (6) is met, and we will design a controller based on Eq. (2). Because of the assumption that $\dot{z}_1$ and $\dot{z}_2$ are taken as the inputs for control u_1 and u_2, Eq. (2) can be rewritten as a basic equation of control:

$$\frac{d}{dt}z = B\boldsymbol{u}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_1 & f_2 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8}$$

Controller Based on Lyapunov Control

In this section, we design a feedback controller based on Lyapunov control that makes the state variables z converge to the desired point, $z = [0, 0, 0]^T$. We introduce the following Lyapunov function:

$$V(z) = \tfrac{1}{2}\left(m_1 z_1^2 + m_2 z_2^2 + m_3 z_3^2\right) \tag{9}$$

where m_1, m_2, and m_3 are positive constants. With Eq. (8), the derivative of $V(t)$ along the motion path is computed as

$$\dot{V} = (\nabla V)^T B\boldsymbol{u} \tag{10}$$

where $\nabla = [\partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3]^T$.

We will consider a class of input vectors expressed as follows:

$$\boldsymbol{u} = -\mathcal{I} B^T \nabla V \tag{11}$$

where $\mathcal{I} \in R^{2\times 2}$. The matrix $\mathcal{I}B^T$ is a tensor that produces an input vector from the gradient vector of the Lyapunov function. Equation (10) can then be expressed in the bilinear form in the gradient vectors,

$$\dot{V} = (\nabla V)^T \mathcal{G} \nabla V \tag{12}$$

where $\mathcal{G} = -B\mathcal{I}B^T \in R^{3\times 3}$. In the following, we will construct the tensor $\mathcal{I}$ by putting a symmetric tensor and an asymmetric tensor together (an extended Lyapunov control method). First, we will investigate how the symmetry of the tensor $\mathcal{I}$ affects the behavior of the controlled system.

Case 1: Symmetric Case

Design the tensor $\mathcal{I}$ as follows:

$$\mathcal{I} = I_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

The corresponding input vector becomes

$$\boldsymbol{u} = -I_s B^T \nabla V \tag{14}$$

This is the case where the controller is designed based on Lyapunov control. Substituting Eq. (14) in Eq. (8), we obtain

$$\dot{z} = -B I_s B^T \nabla V \tag{15}$$

With Eq. (15), the derivative of $V(t)$ is computed as

$$\dot{V} = (\nabla V)^T \mathcal{G}_s \nabla V = -|B^T \nabla V|^2 \leq 0 \tag{16}$$

where $\mathcal{G}_s = -B I_s B^T$. Because of Eqs. (15) and (16), the stable equilibrium points of the system can be seen to be the points on the line $B^T \nabla V = 0$. All of the equilibrium points near the desired point are shown to be stable. By the linearization of Eq. (15) in the neighborhood of the equilibrium points, the stability of the equilibrium points can be determined by the following characteristic equation:

$$\lambda^2 + \alpha_s \lambda + \beta_s = 0 \tag{17}$$

where α_s and β_s are constants depending on each equilibrium point. In the neighborhood of the desired point, α_s and β_s are computed as

$$\alpha_s = m_1 + m_2 + \left(f_1^2 + f_2^2\right)m_3 + \mathcal{O}(z_3) > 0 \tag{18}$$

$$\beta_s = m_1 m_2 + f_1^2 m_2 m_3 + f_2^2 m_1 m_3 + \mathcal{O}(z_3) > 0 \tag{19}$$

Therefore, all of the equilibrium points in the neighborhood of the desired point are stable. Thus, the system may be trapped in a point other than the desired point.

Case 2: Asymmetric Case

Design the tensor $\mathcal{I}$ as follows:

$$\mathcal{I} = \hat{\beta} I_a = \hat{\beta}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{20}$$

where $\hat{\beta}$ is a scalar coefficient. The corresponding input vector becomes

$$\boldsymbol{u} = -\hat{\beta} I_a B^T \nabla V \tag{21}$$

Substituting Eq. (21) in Eq. (8), we obtain

$$\dot{z} = -\hat{\beta} B I_a B^T \nabla V \tag{22}$$

With Eq. (22), the derivative of $V(t)$ is computed as

$$\dot{V} = \hat{\beta}(\nabla V)^T \mathcal{G}_a \nabla V = 0 \tag{23}$$

where $\mathcal{G}_a = -B I_a B^T$. In this case, the state variables z are restricted to a sphere, that is, $V(z) = V_0$, where V_0 is the value of the Lyapunov function determined by the initial state of the system. The equilibrium points of the system in the symmetric case are also equilibrium points in this case. However, in this case, all of the equilibrium points of the system except the desired point may become unstable. To assess their stability, we first linearize Eq. (22) in the neighborhood of these points. Then we can determine the stability of these points by the following characteristic equation:

$$\lambda^2 + \hat{\beta}\alpha_a \lambda + \hat{\beta}^2 \beta_a = 0 \tag{24}$$

where

$$\alpha_a = z_3 m_3 L(z), \qquad \beta_a = \beta_s \tag{25}$$

If $\hat{\beta}$ is chosen such that $\mathrm{sgn}(\hat{\beta}) = -\mathrm{sgn}(\alpha_a)$, the equilibrium points of the system become unstable as long as $\alpha_a \neq 0$. On the other hand, the desired point does not become unstable because $\alpha_a = 0$ at the desired point.

Based on the preceding results, in the following, we will design the tensor $\mathcal{I}$ as follows:

$$\mathcal{I} = \alpha(I_s + \hat{\beta} I_a) \tag{26}$$

where α and $\hat{\beta}$ are scalar coefficients. The input vector becomes

$$\boldsymbol{u} = -\alpha(I_s + \hat{\beta} I_a) B^T \nabla V \tag{27}$$

Coefficient α is a positive constant and affects only the timescale of the motion of the controlled system. Without loss of generality, we can set the coefficient α to unity. The coefficient $\hat{\beta}$ is set to

$$\hat{\beta} = -\beta[\alpha_a / h(g)] \tag{28}$$

where

$$g = |B^T \nabla V| \tag{29}$$

Function $h(g)$ is a smooth function such that

$$h(0) = 0, \qquad h(g) > 0 \quad \text{for} \quad g > 0 \tag{30}$$

Basic equations then become

$$\dot{z} = -B\{I_s - \beta[\alpha_a/h(g)]I_a\}B^T \nabla V \tag{31}$$

where we define $[1/h(g)] \cdot B^T \nabla V = 0$ at $B^T \nabla V = 0$. With Eq. (31), the derivative of $V(t)$ is computed as

$$\dot{V} = (\nabla V)^T \{\mathcal{G}_s - \beta[\alpha_a/h(g)]\mathcal{G}_a\} \nabla V = -|B^T \nabla V|^2 \leq 0 \tag{32}$$

The input vector of Eq. (27) is an extended form of the input vector of Eq. (14) that is designed based on the Lyapunov control; the input vector defined in Eq. (27) contains a symmetric and an asymmetric matrix, and, as a result, the derivative of the Lyapunov function is also composed of a symmetric and an asymmetric bilinear form in the gradient vectors (an extended Lyapunov control method).

In the next section, we will analyze the behavior of systems with this type of controller.

Behavior of the Controlled System

The equilibrium points of the controlled system shown in Eq. (31) are the points on the line $B^T \nabla V = 0$. Because Eq. (32) shows that the controlled system converges to this line, we will examine the stability of the points that lie on it. On this line, the basic equation (31) is discontinuous, and, therefore, to be able to perform the analysis, we modify Eq. (31) as follows:

$$\dot{z} = -B\{I_s - \beta \tanh[h(g)/\epsilon]\,[\alpha_a/h(g)]I_a\}B^T \nabla V \tag{33}$$

Equation (33) approaches Eq. (31) as ϵ is taken to zero. By the linearization of Eq. (33) in the neighborhood of the equilibrium points, the stability of the equilibrium points can be determined using the following characteristic equation:

$$\lambda\{\lambda^2 + [\alpha_s - \beta(\alpha_a^2/\epsilon)]\lambda + [1 + \beta^2(\alpha_a^2/\epsilon^2)]\beta_s\} = 0 \tag{34}$$

The eigenvalues of Eq. (34) are given as

$$\lambda = \Big\{-[\alpha_s - \beta(\alpha_a^2/\epsilon)] \pm \sqrt{[\alpha_s - \beta(\alpha_a^2/\epsilon)]^2 - 4[1 + \beta^2(\alpha_a^2/\epsilon^2)]\beta_s}\Big\}\Big/2, \qquad 0 \tag{35}$$

where parameters α_s, β_s, and α_a are given by Eqs. (18), (19), and (25). The two eigenvectors corresponding to the nonzero eigenvalues lie in the column space of matrix B and determine the stability of the equilibrium points. The condition that the equilibrium points are stable is that the real parts of both eigenvalues are negative, that is,

$$\alpha_s - \beta(\alpha_a^2/\epsilon) > 0, \qquad \beta_s > 0 \tag{36}$$

We rewrite conditions in Eq. (36) as

$$z_3^2 L(z)^2 < \epsilon\alpha_s \big/ \beta m_3^2, \qquad \beta_s > 0 \tag{37}$$

Points on the line $B^T \nabla V = 0$ that satisfy this condition are stable equilibrium points. We consider the region on the line where an equilibrium point is stable:

1) First we consider case 1, the case where $L(z) \neq 0$ at all of the points on the line. In this case, from Eqs. (18) and (19), the points on the line near the origin satisfy $\alpha_s > 0$ and $\beta_s > 0$. On the other hand, only the points on the line near the origin satisfy the first condition of Eq. (37) because ϵ is small and $L(z) \neq 0$. As a result, only the equilibrium points near the origin are stable, and, as ϵ approaches zero, the origin becomes the only stable equilibrium point.

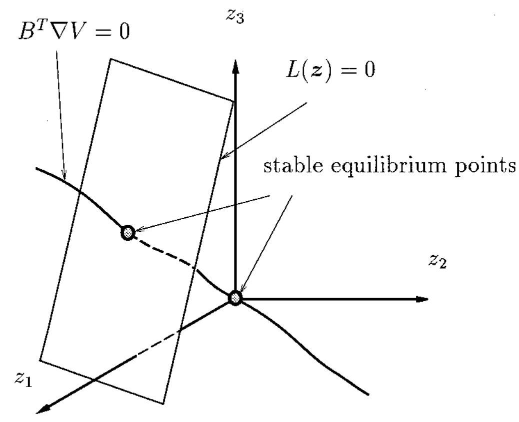

Fig. 1 Stable equilibrium points of the controlled system (case 2, where $\epsilon \to 0$).

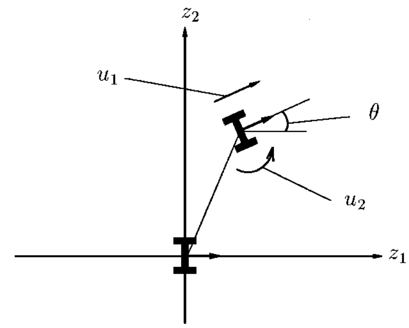

Fig. 2 State variables z_1, z_2, and θ and inputs u_1 and u_2, of a two-wheeled mobile robot.

2) Next we consider case 2, the case where $L(z) = 0$ at some points on the line. In this case, there may be a point where the line $B^T \nabla V = 0$ crosses the plane $L(z) = 0$. If one of such points satisfies the conditions $\alpha_s > 0$ and $\beta_s > 0$, a region of equilibrium points satisfying condition (37) exists near that equilibrium point because the value of $L(z)$ is small. All of the equilibrium points in the region are stable. From Eq. (37), as the parameter ϵ approaches zero, the region shrinks until only the point where the line $B^T \nabla V = 0$ crosses the plane $L(z) = 0$ becomes the stable equilibrium point. As a result, in this case, there may exist stable equilibrium points besides the origin, where the system is not controllable (Fig. 1).

From the preceding analysis, when the system satisfies the controllability criterion at all of the equilibrium points, the controlled system converges to the desired point. However, when equilibrium points where the system does not satisfy the controllability criterion exist, the controlled system may be trapped in an equilibrium point that is not the desired point.

Example: Two-Wheeled Mobile Robot

In this section we apply the extended Lyapunov control method to an example, a two-wheeled mobile robot, and check the behavior of the system by numerical simulations. We set the parameters of the Lyapunov function given in Eq. (9), m_1, m_2, and m_3, to 1.0.

As shown in Fig. 2, the state of a two-wheeled mobile robot can be expressed in terms of the position of the mobile robot on the plane z_1 and z_2 and the angle θ between the current direction of the mobile robot and the positive direction of the z_1 axis. The inputs of the system are translational velocity u_1 and angular velocity u_2. The desired point is set to be the origin. If the wheels do not skid at all, the motion of the mobile robot can be determined by the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{38}$$

This system satisfies the condition of the controllability $L(z) \neq 0$ at all points.

Analysis

The inputs are assigned as follows:

$$u_1 = -(z_1 \cos\theta + z_2 \sin\theta) - \beta[z_2/h(g)]\theta \quad (39)$$

$$u_2 = -\theta + \beta[z_2/h(g)](z_1 \cos\theta + z_2 \sin\theta) \quad (40)$$

The behavior of the system is analyzed for the following two cases: case 1, $h(g) = g$, and case 2, $h(g) = g^2$.

Case 1

The basic equation, Eq. (31), corresponding to Eq. (38), is expressed as

$$\dot{z}_1 = -\cos\theta(z_1 \cos\theta + z_2 \sin\theta) - \beta(z_2/g)\theta \cos\theta \quad (41)$$

$$\dot{z}_2 = -\sin\theta(z_1 \cos\theta + z_2 \sin\theta) - \beta(z_2/g)\theta \sin\theta \quad (42)$$

$$\dot{\theta} = -\theta + \beta(z_2/g)(z_1 \cos\theta + z_2 \sin\theta) \quad (43)$$

where

$$g = |B^T \nabla V| = \sqrt{(z_1 \cos\theta + z_2 \sin\theta)^2 + \theta^2} \quad (44)$$

From Eqs. (41–43), the variable g satisfies the following equation,

$$\dot{g} = -g + \beta\left(z_2\big/g^2\right)(z_2 \cos\theta - z_1 \sin\theta)(z_1 \cos\theta + z_2 \sin\theta)^2$$
$$- (\theta/g)(z_2 \cos\theta - z_1 \sin\theta)(z_1 \cos\theta + z_2 \sin\theta) \quad (45)$$

Because the variables z_1, z_2, θ, and g decrease with time, we can assume in the following that they are small.

1) First we examine the behavior of the system in the region $g > \mathcal{O}(\beta z_2^2)$. Neglecting the small terms, we obtain the equations for z_2 and g as

$$\dot{z}_2 = 0$$

$$\dot{g} = -g + \beta\left(z_2\big/g^2\right)(z_2 \cos\theta - z_1 \sin\theta)(z_1 \cos\theta + z_2 \sin\theta)^2$$
$$< -g + \beta z_2^2 < 0 \quad (46)$$

Equations (46) show that the system moves into the region $g \leq \mathcal{O}(\beta z_2^2)$.

2) Next, we examine the behavior of the system in the region $g \leq \mathcal{O}(\beta z_2^2)$. When the small terms are neglected, Eqs. (41–43) and (45) become

$$\dot{z}_1 = -\beta(z_2/g)\theta, \qquad \dot{z}_2 = 0, \qquad \dot{\theta} = \beta(z_2/g)z_1, \qquad \dot{g} = 0 \quad (47)$$

where variable g is approximately expressed as $g = \sqrt{(z_1^2 + \theta^2)}$. Equations (47) show that the variables z_2 and g are constant whereas the variables z_1 and θ oscillate. The variables z_1 and θ are expressed as

$$z_1 = g\cos(\omega_c t + \phi), \qquad \theta = g\sin(\omega_c t + \phi) \quad (48)$$

where $\omega_c = \beta z_2/g$ and ϕ is a constant. Substituting Eqs. (48) into Eqs. (42) and (45), averaging them with the period $2\pi/\omega_c$, and neglecting smaller terms, we obtain

$$\dot{z}_2 = -(\beta/2)g z_2 \quad (49)$$

$$\dot{g} = -g + \beta z_2^2\big/2 = -g + g_c \quad (50)$$

where

$$g_c(z_2) = (\beta/2)z_2^2 \quad (51)$$

Equations (49) and (50) show that $|\dot{z}_2| \ll |\dot{g}|$. Therefore, variable g converges to g_c exponentially, whereas variable z_2 is constant. Substituting steady-state solution g_c of variable g into Eq. (49), we obtain

$$\dot{z}_2 = -(\beta^2/4)z_2^3 \quad (52)$$

From Eq. (52), we have

$$|z_2| = 1\Big/\sqrt{(\beta^2/2)t + C} \quad (53)$$

where C is a constant. By the substitution of Eq. (53) into Eq. (51), the corresponding solution of variable g is expressed as

$$g = \beta/(\beta^2 t + 2C) \quad (54)$$

By the use of Eqs. (53) and (54), variables z_1 and θ are expressed as

$$z_1 = [\beta/(\beta^2 t + 2C)]\cos\left(\sqrt{2\beta^2 t + 4C}\,t + \phi\right)$$

$$\theta = [\beta/(\beta^2 t + 2C)]\sin\left(\sqrt{2\beta^2 t + 4C}\,t + \phi\right) \quad (55)$$

Expressions (55) show that the amplitude of oscillation decreases as $\mathcal{O}(t^{-1})$, whereas the frequency of oscillation increases as $\mathcal{O}(t^{1/2})$. On the other hand, input $\boldsymbol{u}$ decreases as $\mathcal{O}(t^{-1/2})$ and tends to zero as $t \to \infty$.

Case 2

The basic equation, Eq. (31), corresponding to Eq. (38), is expressed as

$$\dot{z}_1 = -\cos\theta(z_1 \cos\theta + z_2 \sin\theta) - \beta\left(z_2\big/g^2\right)\theta \cos\theta \quad (56)$$

$$\dot{z}_2 = -\sin\theta(z_1 \cos\theta + z_2 \sin\theta) - \beta\left(z_2\big/g^2\right)\theta \sin\theta \quad (57)$$

$$\dot{\theta} = -\theta + \beta\left(z_2\big/g^2\right)(z_1 \cos\theta + z_2 \sin\theta) \quad (58)$$

From Eqs. (56–58), the variable g satisfies the following equation:

$$\dot{g} = -g + \beta\left(z_2\big/g^3\right)(z_2 \cos\theta - z_1 \sin\theta)(z_1 \cos\theta + z_2 \sin\theta)^2$$
$$- (\theta/g)(z_2 \cos\theta - z_1 \sin\theta)(z_1 \cos\theta + z_2 \sin\theta) \quad (59)$$

We also assume in the following that the variables z_1, z_2, θ, and g are small:

1) First we examine the behavior of the system in the region $g^2 < \mathcal{O}(\beta|z_2|)$. In this region, when the small terms are neglected, Eqs. (56–59) become

$$\dot{z}_1 = -\beta\left(z_2\big/g^2\right)\theta, \qquad \dot{z}_2 = 0, \qquad \dot{\theta} = \beta\left(z_2\big/g^2\right)z_1, \qquad \dot{g} = 0 \quad (60)$$

Equations (60) show that the variables z_2 and g are constant, whereas the variables z_1 and θ oscillate and are expressed as

$$z_1 = g\cos(\omega_{c2} t + \phi_2), \qquad \theta = g\sin(\omega_{c2} t + \phi_2) \quad (61)$$

where $\omega_{c2} = \beta z_2/g^2$ and ϕ_2 is a constant. Substituting Eqs. (61) into Eqs. (57) and (59), averaging them with the period $2\pi/\omega_{c2}$, and neglecting smaller terms, we obtain

$$\dot{z}_2 = -(\beta/2)z_2 \quad (62)$$

$$\dot{g} = -g + \beta z_2^2\big/2g \quad (63)$$

From Eq. (62), we have

$$z_2 = C_1 e^{-(\beta/2)t} \quad (64)$$

Substituting Eq. (64) into Eq. (63), we obtain

$$\text{for}\quad \beta \neq 2, \qquad g = \sqrt{C_2 e^{-2t} + [\beta/(2-\beta)]C_1^2 e^{-\beta t}}$$

$$\text{for}\quad \beta = 2, \qquad g = \sqrt{\left(\beta C_1^2 t + C_2\right)e^{-2t}} \quad (65)$$

where C_1 and C_2 are constant. When $\beta < 4.0$, the system moves within the region $g^2 < \mathcal{O}(\beta|z_2|)$ for all $t > 0$ and converges to the origin. On the contrary, when $\beta \geq 4.0$, the system cannot satisfy the condition that $g^2 < \mathcal{O}(\beta|z_2|)$ for all $t > 0$ and may go toward the region where $g^2 \geq \mathcal{O}(\beta|z_2|)$ with time. In this region, $g^2 < \mathcal{O}(\beta|z_2|)$, the inputs u_1 and u_2 behave as follows: Because the amplitude of oscillation of the inputs is $\mathcal{O}(\beta z_2/g)$ from Eqs. (39) and (40), as time advances it approaches a constant $\sqrt{[\beta(2-\beta)]}$ if $\beta < 2.0$ and approaches 0 if $\beta \geq 2.0$. On the other hand, because the frequency of oscillation of the inputs is $\mathcal{O}(\beta z_2/g^2)$ from Eqs. (39) and (40), it increases exponentially if $\beta < 4.0$ and decreases exponentially if $\beta > 4.0$ with time.

2) Next, we examine the behavior of the system in the region $g^2 \geq \mathcal{O}(\beta|z_2|)$. Neglecting the small terms, we obtain the equation for g as

$$\dot{g} = -g \tag{66}$$

From Eq. (66), we obtain

$$g = C_3 e^{-t} \tag{67}$$

where C_3 is constant. When $\beta \geq 4.0$, as stated previously, the system moves from the region $g^2 < \mathcal{O}(\beta|z_2|)$ to this region, $g^2 \geq \mathcal{O}(\beta|z_2|)$. Then, as a result of Eq. (67), the system converges to the origin in this region. On the contrary, when $\beta < 4.0$, according to the initial value of the system, there are cases where the system stays in this region or moves to the region $g^2 < \mathcal{O}(\beta|z_2|)$. However, in both cases, the system converges to the origin. In this region, because $\beta|z_2| \leq \mathcal{O}(e^{-2t})$, the magnitude of the inputs u_1 and u_2 converges to 0 using Eqs. (39) and (40).

The motion of the system for the case where $h(g) = g^2$ is summarized as follows: The behavior of the system is different according to the value of the parameter β and initial conditions. However, the system converges to the origin and the magnitude of the inputs approaches 0 or a constant with time. Note that if we set the parameter β such that $\beta \geq 4.0$, the system converges to the origin exponentially, the magnitude of the inputs converges to 0, and oscillation of the inputs with a high frequency can be avoided.

Numerical Simulations

Case 1

Numerical simulations were executed for the following two cases, case 1a and case 1b (Table 1). In case 1a, the value of the parameter β is set to 1.0. Figure 3 shows the behavior of the system in the $z_1 z_2$ plane. Figure 4 shows the time history of variable g. The dashed line in Fig. 4 is a line proportional to t^{-1} as shown in Eq. (54). Variable g decreases along this line, as the analysis has revealed. In case 1b, the value of the parameter β is set to 10.0. Figure 5 shows the motion of the system in the $z_1 z_2$ plane. In this case, the system moves to the neighborhood of the desired point after three large switchbacks, and in the neighborhood of the origin, the system behaves in the same way as in case 1a.

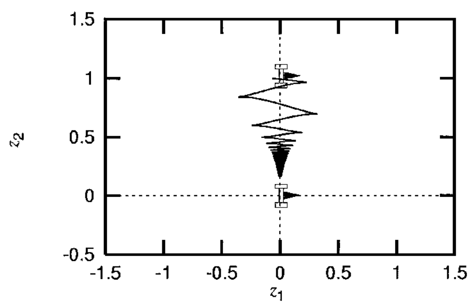

Fig. 3 Motion of the system in the $z_1 z_2$ plane (case 1a).

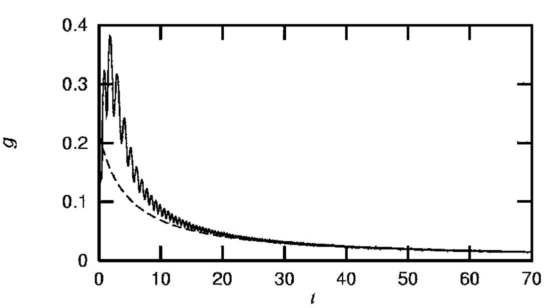

Fig. 4 Time history of g (case 1a).

Table 1 Simulation cases

Case	$h(g)$	β	Initial value of z^T
1a	g	1.0	(0.01, 1.0, 0.01)
1b	g	10.0	(0.01, 1.0, 0.01)
2a	g^2	1.0	(0.5, 1.0, 0.0)
2b	g^2	5.0	(0.5, 1.0, 0.0)

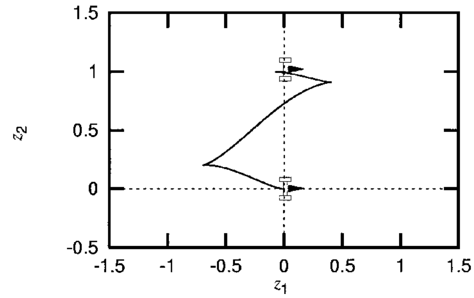

Fig. 5 Motion of the system in the $z_1 z_2$ plane (case 1b).

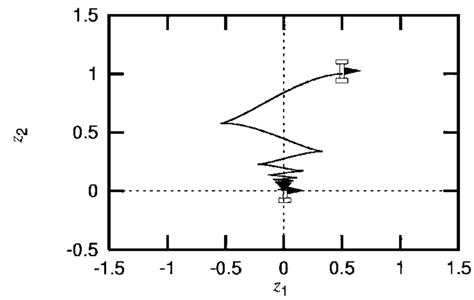

Fig. 6 Motion of the system in the $z_1 z_2$ plane (case 2a).

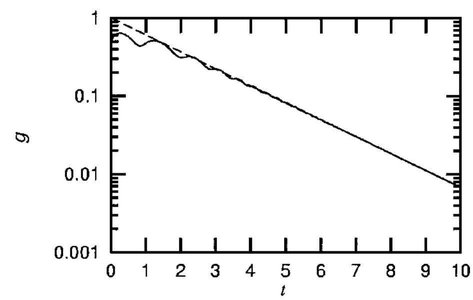

Fig. 7 Time history of g (case 2a).

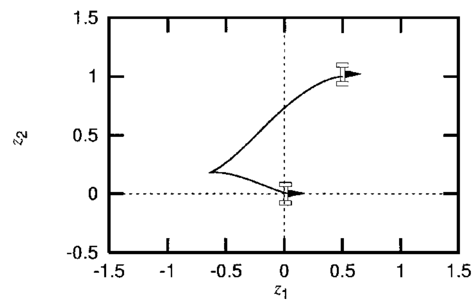

Fig. 8 Motion of the system in the $z_1 z_2$ plane (case 2b).

Case 2

We executed numerical simulations for the following two cases, case 2a, where the system moves within the region $g^2 < \mathcal{O}(\beta|z_2|)$, and case 2b, where the system moves from the region $g^2 < \mathcal{O}(\beta|z_2|)$ to the region $g^2 \geq \mathcal{O}(\beta|z_2|)$. In case 2a, the value of the parameter β is set to 1.0 (see Table 1). Figure 6 shows the behavior of the system in the $z_1 z_2$ plane. Figure 7 shows the time history of variable g. The dashed line in Fig. 7 is a line proportional to $e^{-t/2}$ as shown

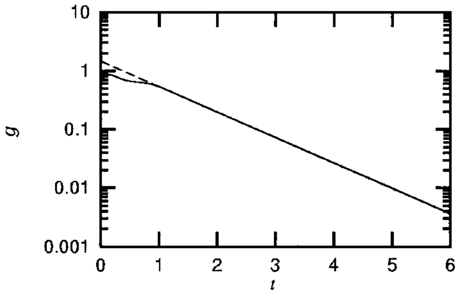

Fig. 9 Time history of g (case 2b).

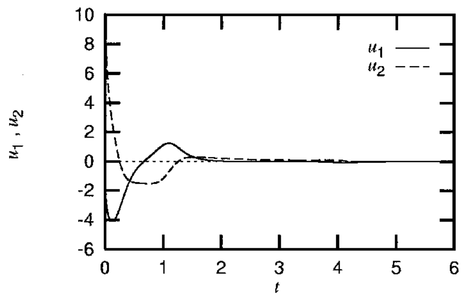

Fig. 10 Time histories of u_1 and u_2 (case 2b).

in Eq. (65). In case 2b, the value of the parameter β is set to 5.0. Figure 8 shows the motion of the system in the $z_1 z_2$ plane. In this case, the system moves to the neighborhood of the desired point after two large switchbacks and, in the neighborhood of the origin, converges to the origin exponentially without the oscillation of the variables z_1 and θ. Figure 9 shows the time history of variable g, which matches the solution given in Eq. (67) very well. Note that the inputs u_1 and u_2 converge to zero and do not oscillate with a high frequency, as shown in Fig. 10.

Conclusions

In this paper, we proposed a discontinuous state feedback control law for a three-dimensional two-input nonholonomic system without drift, based on an extended Lyapunov control method. When the system satisfies the condition of the controllability at all of the equilibrium points, the proposed controller makes the system converge to the desired point. This is verified by analysis and numerical simulations of a two-wheeled mobile robot. From a practical point of view, note that, by choosing suitable control parameters, the controlled system can be made to converge exponentially to the desired point, and the inputs can decay to zero without oscillations.

References

[1] Isidori, A., *Nonlinear Control Systems*, 2nd ed., Springer-Verlag, Berlin, 1989, pp. 243–259.

[2] Brockett, R. W., "Asymptotic Stability and Feedback Stabilization," *Differential Geometric Control Theory*, edited by R. W. Brockett, R. S. Millman, and H. J. Sussman, Birkhauser, Boston, 1983, pp. 181–191.

[3] Samson, C., "Control of Chained Systems Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots," *IEEE Transactions on Automatic Control*, Vol. 40, No. 1, 1995, pp. 64–77.

[4] Pomet, J.-B., "Explicit Design of Time-Varying Stabilizing Control Laws for a Class of Controllable Systems Without Drift," *System and Control Letters*, Vol. 18, No. 2, 1992, pp. 147–158.

[5] Sørdalen, O. J., and Egeland, O., "Exponential Stabilization of Nonholonomic Chained Systems," *IEEE Transactions on Automatic Control*, Vol. 40, No. 1, 1995, pp. 35–49.

[6] M'Closkey, R. T., and Murray, R. M., "Exponential Stabilization of Driftless Nonlinear Control Systems Using Homogeneous Feedback," *IEEE Transactions on Automatic Control*, Vol. 42, No. 5, 1997, pp. 614–628.

[7] Khennouf, H., and Canudas de Wit, C., "Quasi-Continuous Exponential Stabilizers for Nonholonomic Systems," *Proceedings of International Federation of Automatic Control 13th World Congress*, Vol. F, Elsevier Science, Oxford, England, U.K., 1996, pp. 49–54.

[8] Astolfi, A., "Discontinuous Control of Nonholonomic Systems," *System and Control Letters*, Vol. 27, No. 1, 1996, pp. 37–45.

[9] Astolfi, A., "Exponential Stabilization of a Wheeled Mobile Robot via Discontinuous Control," *Journal of Dynamic Systems Measurement and Control*, Vol. 121, No. 1, 1999, pp. 121–125.

[10] Lafferriere, G. A., and Sontag, E. D., "Remarks on Control Lyapunov Functions for Discontinuous Stabilizing Feedback," *Proceedings of the 32nd Conference on Decision and Control*, Inst. of Electrical and Electronics Engineers, New York, 1993, pp. 306–308.

[11] Bryant, R. L., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., and Griffiths, P. A., *Exterior Differential Systems*, Springer-Verlag, New York, 1991, pp. 27–43.